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SEMILINEAR EVOLUTION EQUATIONS IN BANACH SPACES WITH APPLICATION TO PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. A theory for a class of semilinear evolution equations in Banach spaces is developed which when applied to certain parabolic partial differential equations with nonlinear terms in divergence form gives strong solutions even for nondifferentiable data.

1. INTRODUCTION

The class of equations considered in this work have the form

$$u'(t) + Au(t) = F(u(t)), \qquad t > 0, \tag{1}$$

$$u(0) = \phi. \tag{2}$$

We consider (1)(2) as a Cauchy problem in a Banach space X; A is a closed linear operator which is densely defined and $-A$ generates an analytic semigroup $T(t)$: $t \geq 0$. Many researchers have studied equations similar to (1)(2) in the last fifteen years. The books by Friedman [3], Henry [5], and Pazy [7] give good accounts of the most important results. The part of the theory that we are most interested in, is when the operator A has fractional powers and the nonlinear operator F maps from a fractional power space into X. Of course in the references cited much has been written in this regard. However, the course taken here is to exploit special structure which may occur. To be precise we consider nonlinear operators F which can be written in the form $A^\alpha G$ where G maps a subspace Y of X into X.

The motivation for an abstract theory such as this comes from the following partial differential equation:

$$w_t(x, t) - w_{xx}(x, t) = \sigma(w(x, t))_x, \tag{3}$$

$$w(0, t) = w(1, t) = 0, \qquad t > 0, \tag{4}$$

$$w(x, 0) = s(x), \qquad 0 < x < 1. \tag{5}$$

It is not true in general that $\partial/\partial x = A^{1/2}$, however we will show in §4 for a special class of operators A there exist a bounded linear operator B from X

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into X such that $A^{1/2}B = \partial/\partial x$. Letting $G = B\sigma$ we can fit the equations (3), (4), (5), into the abstract theory developed in this paper.

The abstract theory one can find in the books [3, 5, 7] handles partial differential equations like (3)(4)(5), however the theory illustrated in these works does not distinguish between the problems of the form

$$w_t(x, t) - w_{xx}(x, t) = w(x, t)|w(x, t)|^{\beta-1} \tag{6}$$

with initial and boundary conditions and equations of the form of (6) but with right-hand side

$$\frac{\partial}{\partial x}\{w(x, t)|w(x, t)|^{\beta-1}\}. \tag{7}$$

Weissler [10] has studied problems illustrated by (6) and has proved existence in appropriate L_p spaces. We show here that solutions of (6) with right-hand side (7) can be solved in an L_p space. This is quite different from the theory presented in Pazy [7] and others where both equations (6) and (7) would have been solved in appropriate Sobolev spaces. Roughly speaking, the theory developed here can be applied to partial differential equations with right-hand sides in divergence form. Taking advantage of this form and the analyticity of the semigroup generated by the linear part, we can show that starting with initial data in an appropriate L_p space a weak solution in this space can be found and with additional assumptions show this weak solution is a strong solution for $t > 0$. In §4 we see how the theory developed in this paper can be applied to equations such as the generalized Burger's equation, the Cahn-Hilliard equation, and the Navier-Stokes equation.

2. Assumptions

The following assumptions will be used throughout:

(i) X and Y are Banach spaces with $Y \subset X$.

(ii) $-A$ generates an analytic semigroup of linear operators, $T(t)$, $t \geq 0$ on X.

(iii) $A^{-\alpha}$ for $0 < \alpha < 1$ is defined by the integral

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} T(s)\, ds,$$

where $\Gamma(\alpha)$ denotes the gamma function. Then $A^\alpha \equiv (A^{-\alpha})^{-1}$ exists as a closed and densely defined linear operator. $A^{-\alpha}: X \to D(A^\alpha)$ is a bounded linear operator, where $D(A^\alpha)$ is the domain of A^α. Denote X_α as the Banach space created by norming $D(A^\alpha)$ with the graph norm.

(iv) There exist $M \geq 1$ and $\omega \in R$ such that for all $\alpha \geq 0$, $\|T(t)\|_X \leq Me^{\omega t}$ and $\|A^\alpha T(t)\|_X \leq C_\alpha t^{-\alpha} e^{\omega t}$ for some $C_\alpha > 0$.

We shall assume $X_\alpha \subseteq Y \subseteq X$ so that $T(t): X \to Y$ for all $t > 0$ is a bounded linear operator and

(v) $T(t)y$ is continuous in t in the norm of Y for each $y \in Y$.

(vi) $A^\beta T(t) = T(\frac{t}{2}) A^\beta T(\frac{t}{2}): X \to Y$, for $t > 0$ and $\|A^\beta T(t)\|_{X\to Y} \in L^1(0, r)$ for $\beta \in [\alpha, \alpha + d]$ for some $d > 0$ and every $r > 0$.

(vii) The function F maps X_α into X, and satisfies: there exists $G: Y \to X$ such that G is locally Lipschitz, $G: X_\alpha \to X_\alpha$ and for each $\phi \in X_\alpha$, $F(\phi) = A^\alpha G(\phi)$.

It is known, for example Pazy [7], that equation (1) is related to the integral equation

$$u(t) = T(t)\phi + \int_0^t T(t-s)F(u(s))\,ds, \qquad t \geq 0, \tag{8}$$

where $T(t)$ is the semigroup of operators generated by $-A$. The solution $u(t)$ of equation (8) is called a mild solution of equation (1) and is not necessarily a solution of equation (1). With some additional regularity properties it can be shown that a solution of equation (8) is a solution of equation (1).

Definition. *Mild solution.* A function $u(t)$ is a mild solution of equation (1)(2) on $[0, T)$ if $u \in C([0, T); X_\alpha)$, $u(0) = \phi$, and $u(t)$ satisfies the integral equation (8) on $[0, T)$.

Definition. *Strong solution.* A function $u(t)$ is a strong solution of equation (1)(2) if $u(0) = \phi$, $u \in C([0, T); X_1) \cap C^1([0, T); X)$, and $u(t)$ satisfies (1)(2) on $[0, T)$.

Operators generated by strongly elliptic partial differential equations defined on bounded domains in R^n with smooth boundaries generate semigroups of linear operators with the properties outlined above. This will be illustrated in §4.

3. Main results

In this section local existence and a continuation result for an equation related to (8) is proved. It is then shown that with some additional assumptions this solution is a solution to equation (8) and also equation (1).

Theorem 1. *Let the assumptions* (i) *through* (vii) *hold, then for each* $\phi \in Y$ *there exists a* $T > 0$ *and a unique continuous function* $u\colon [0, T) \to Y$ *such that*

$$u(t) = T(t)\phi + \int_0^t A^\alpha T(t-s)G(u(s))\,ds, \qquad t \geq 0. \tag{9}$$

Proof. Define the set $\Delta = \{u\colon [0, t] \to Y \mid u(t)$ is continuous, $u(0) = \phi$ and $\sup_{0\leq t\leq T} \|u(t) - \phi\|_Y \leq R\}$. Further define the mapping L on Δ by

$$(Lu)(t) = T(t)\phi + \int_0^t A^\alpha T(t-s)G(u(s))\,ds.$$

The mapping L is a contraction of Δ into itself. First note that for $u \in \Delta$, L is well defined since

$$\int_0^t \|A^\alpha T(t-s)G(u(s))\|\,ds \leq \left\{\int_0^t \|A^\alpha T(t-s)\|_{X\to Y}\,ds\right\} \times \{\|G(\phi)\|_X + KR\}$$

where for $\phi_1, \phi_2 \in \{z\colon \|z - \phi\|_Y \leq R\}$,

$$\|G(\phi_1) - G(\phi_2)\|_X \leq K\|\phi_1 - \phi_2\|_Y.$$

It is also true that

$$\|T(t)\phi + \int_0^t A^\alpha T(t-s)G(u(s))\,ds - \phi\|_Y \leq \|T(t)\phi - \phi\|_Y + \left\{\int_0^t \|A^\alpha T(t)\|_{X\to Y}\,ds\right\} \times \{\|G(\phi)\|_X + KR\}$$

for each $u \in \Delta$. The right side of the above inequality can be made less than R by choosing $T > 0$ small enough. This proves $L(\Delta) \subseteq \Delta$. Let $u, v \in \Delta$, then we have

$$\begin{aligned}
&\|(Lu)(t) - (Lv)(t)\|_Y \\
&\quad = \left\| \int_0^t A^\alpha T(t-s)[G(u(s)) - G(v(s))]\,ds \right\|_Y \\
&\quad \le \int_0^t \|A^\alpha T(t-s)[G(u(s)) - G(v(s))]\,ds\|_Y \\
&\quad \le \int_0^t \|A^\alpha T(t-s)\|_{X\to Y} \|G(u(s)) - G(v(s))\|_X\,ds \\
&\quad \le \left\{ K \int_0^t \|A^\alpha T(s)\|_{X\to Y}\,ds \right\} \times \left\{ \sup_{0\le t\le T} \|u(t) - v(t)\|_Y \right\}.
\end{aligned}$$

The quantity $K\int_0^t \|A^\alpha T(s)\|_{X\to Y}\,ds$ can be made less than one by again restricting T.

Remark. With further assumptions on $A^\alpha T(t)$, $t > 0$, F, and X_α, it can be shown that the solution $u(t)$ of equation (9) guaranteed by Theorem 1 is also a solution to equation (1). To this end we establish the following lemmas.

Lemma 1. *Let the conditions* (i) *through* (vii) *hold and suppose* $\|A^\alpha T(t)\|_{X\to Y} \le Ct^{-\zeta}$ *for* $t \in (0, T]$, $0 < \zeta < 1$ *for some* $C > 0$, *then for all* $\gamma > 0$ *and* $t, t+h \in [\gamma, T]$ *there exist* σ *such that*

$$\|u(t+h) - u(t)\|_Y \le C(\gamma)h^\sigma, \qquad 0 < \sigma < 1.$$

Proof. We prove the case $t + h > t$. We have then

$$\begin{aligned}
\|u(t+h) - u(t)\|_Y &\le \|(T(h) - I)T(t)\phi\|_Y \\
&\quad + \left\| \int_0^t A^\alpha[(T(h) - I)T(t-s)]G(u(s))\,ds \right\|_Y \\
&\quad + \left\| \int_t^{t+h} A^\alpha T(t+h-s)G(u(s))\,ds \right\|_Y \\
&\le \|A^\alpha T(t)(T(h) - I)A^{-\alpha}\phi\|_Y \\
&\quad + \left\| (T(h) - I) \int_0^t A^\alpha T(t-s)Gu(s))\,ds \right\|_Y \\
&\quad + \left\| \int_t^{t+h} A^\alpha T(t+h-s)G(u(s))\,ds \right\|_Y \\
&\le \frac{C}{\gamma^\zeta} \|(T(h) - I)A^{-\alpha}\phi\|_X + C(\|G(\phi)\|_X + KR)h^{1-\zeta} \\
&\quad + \int_0^t \|A^{\alpha+\varepsilon}T(t-s)[T(h) - I]A^{-\varepsilon}G(u(s))\,ds\|_Y \\
&\le \frac{C}{\gamma^\zeta} \|\phi\|_X h^\alpha + C(\|G(\phi)\| + KR)h^{1-\zeta} \\
&\quad + \left[C(\|G(\phi)\|_X + KR) \int_0^T \|A^{\alpha+\varepsilon}T(s)\|_{X\to Y}\,ds \right] h^\varepsilon
\end{aligned}$$

where $0 < \varepsilon < d$. Letting $\sigma = \min\{\alpha, 1-\zeta, \varepsilon\}$ the lemma is proved.

This proves that a solution of equation (9) is locally Hölder continuous in the Y norm on $(0, T]$. If the solution $u(t)$ of equation (9) is in X_α and if it is also Hölder continuous in the X_α norm we can show that $u(t)$ is a solution of (1) if F is locally Lipschitz continuous from X_α into X.

Lemma 2. *Assume the conditions of Lemma* 1 *hold, then the solution* $u(t)$ *of equation* (9) *is in* $X_{1-\alpha}$ *for* $t \in (0, T]$.

Proof. Let $\gamma > 0$, then the solution $u(t)$ of equation (9) satisfies

$$u(t) = T(t-\gamma)u(\gamma) + \int_\gamma^t A^\alpha T(t-s)G(u(s))\,ds$$

and

$$u(t) = T(t-\gamma)u(\gamma) + \int_\gamma^t A^\alpha T(t-s)G(u(t))\,ds + \int_\gamma^t A^\alpha T(t-s)[G(u(s)) - G(u(t))]\,ds.$$

Since $T(t-\gamma)u(\gamma) \in D(A)$ for all $t > \gamma$ and $\int_\gamma^t A^\alpha T(t-s)G(u(t))\,ds = A^{\alpha-1}[G(u(t)) - T(t-\gamma)G(u(t))]$ for all $t \geq \gamma$. We have only to check that $\int_\gamma^t A^\alpha T(t-s)[G(u(s)) - G(u(t))]\,ds$ is in $X_{1-\alpha}$. To this end we note that

$$\begin{aligned}\left\|A^{1-\alpha}\int_\gamma^t A^\alpha T(t-s)[G(u(s)) - G(u(s))]\,ds\right\|_X &= \left\|\int_\gamma^t A^1 T(t-s)[G(u(s)) - G(u(t))]\,ds\right\|_X \\ &\leq CK\int_\gamma^t (t-s)^{-1}(t-s)^\sigma\,ds = \frac{CK}{\varepsilon}(t-\gamma)^\sigma.\end{aligned}$$

The last inequality is true from the results of Lemma 1.

Lemma 3. *Assume that the conditions of the previous lemmas hold and that* $X_{1-\alpha} \subseteq X_\alpha$, *the imbedding being continuous, then the solution* $u(t)$ *of equation* (9) *is a mild solution of equation* (1) *and is locally Hölder continuous into* X_α.

Proof. Lemma 2 and the assumption $X_{1-\alpha} \subseteq X_\alpha$ implies that $u(t) \in X_\alpha$ for all $t > 0$. Thus for $\gamma > 0$ and $t > \gamma/2$ we have

$$u(t) = T\left(t - \frac{\gamma}{2}\right)u\left(\frac{\gamma}{2}\right) + \int_{\gamma/2}^t T(t-s)F(u(s))\,ds. \tag{10}$$

Since $u(t)$ is continuous into $X_{1-\alpha}$ and $X_{1-\alpha}$ is continuously imbedded into X_α, $u(t)$ is continuous into X_α. To show that $u(t)$ is locally Hölder continuous into X_α let $\gamma > 0$ and $t+h$, t be in $[\gamma, T]$ then

$$\begin{aligned}\|u(t+h)-u(t)\|_{X_\alpha} &\le \left\|A^\alpha(T(h)-I)T\left(t-\frac{\gamma}{2}\right)u\left(\frac{\gamma}{2}\right)\right\|_X \\ &\quad+\left\|\int_{\gamma/2}^{t} A^\alpha T(t-s)F(u(s))\,ds\right\|_X \\ &\quad+\left\|\int_{\gamma/2}^{t+h} A^\alpha T(t+h-s)F(u(s))\,ds\right\|_X \\ &\le \left\|(T(h)-I)A^\alpha T\left(t-\frac{\gamma}{2}\right)u\left(\frac{\gamma}{2}\right)\right\|_X \\ &\quad+\left\|\int_{\gamma/2}^{t}(T(h)-I)A^\alpha T(t-s)F(us(s))\,ds\right\|_X \\ &\quad+\left\|\int_{t}^{t+h} A^\alpha T(t+h-s)F(u(s))\,ds\right\|_X \\ &\le C\left\|u\left(\frac{\gamma}{2}\right)\right\|_{X_\alpha} h^\alpha+\frac{C}{1-\alpha}\sup_{\gamma/2\le t\le T}\|F(u(t))\|_X h^{1-\alpha} \\ &\quad+\left\{C\int_{\gamma/2}^{t}\|A^{\alpha+\varepsilon}T(t-s)\|_X\,ds\right\}\times\left\{\sup_{\gamma/2\le t\le T}\|F(u(s))\|_X\right\}h^\varepsilon\end{aligned}$$

where ε is chosen so that $\alpha+\varepsilon<1$. Thus there exist a $C>0$ and a $0<\mu<1$ such that

$$\|u(t+h)-u(t)\|_{X_\alpha}\le Ch^\mu$$

for $t+h$, $t\in[\gamma, T]$.

Theorem 2. *Assume the conditions* (i) *through* (vii), *the assumptions of Lemma* 3, *and that* F *is locally Lipshitz from* X_α *into* X, *then any solution of equation* (9) *is also a strong solution of equation* (1) *for* $t>0$.

Proof. Since F is locally Lipshitz and $u(t)$ is locally Hölder continuous into X_α, the function $F(u(t))$ is locally Hölder continuous on $[\gamma, T]$ for any $\gamma>0$. Thus the theory of analytic semigroups of linear operators Pazy [7] gives the desired result.

The next result gives information about continuation of solutions of equation (9). This result says "in practice" that continuation of solutions depends on a bound in the Y norm. For applications to partial differential equations this could mean an L_p bound rather than a bound in an appropriate Sobolev space. Once a bound is obtained, a global regular solution can be obtained from Theorem 2 and the representation (8).

Theorem 3. *Let the conditions of Lemma* 1 *hold, then* $u(t)$ *may be extended to a maximum interval of existence* $[0, T_{\max})$. *If* $T_{\max}<\infty$ *then*

$$\lim_{t\to T_{\max}}\int_0^t\|A^\alpha T(t-s)\|_{X\to Y}\|G(u(s))\|_X=\infty \tag{11}$$

and

$$\lim_{t\to T_{\max}}\|u(t)\|_Y=\infty. \tag{12}$$

Proof. The existence of a maximally defined interval $[0, T_{\max})$ follows in the usual way from the uniqueness of solutions. Suppose $T_{\max} < \infty$. For each $t \in [0, T_{\max})$ $u(t)$ satisfies the integral equation (9). Claim $\lim_{t\to T_{\max}} \sup\|u(t)\|_Y \le C$ for all $t \in [0, T_{\max})$ and some $C > 0$. This implies

$$\|u(t)\|_Y \le \|T(t)\phi\|_Y + \left\{\int_0^{T_{\max}} \|A^\alpha T(t-s)\|_{X\to Y}\, ds\right\} \times \left\{\sup_{0\le t<T_{\max}} \|G(u(t))\|_X\right\}.$$

Now for $0 < \tau < t < T_{\max}$ we have

$$\begin{aligned} \|u(t)-u(\tau)\|_Y &\le \|A^\alpha T(\tau)[T(t-\tau)-I]A^{-\alpha}\phi\|_Y \\ &\quad + \left\|\int_0^\tau A^{\alpha+\varepsilon}T(\tau-s)[T(t-\tau)-I]A^{-\varepsilon}G(u(s))\,ds\right\|_Y \\ &\quad + \left\|\int_\tau^t A^\alpha T(t-s)G(u(s))\,ds\right\|_Y \\ &\le \frac{C}{\tau^\zeta}\|\phi\|_Y(t-\tau)^\alpha + C(t-\tau)^\varepsilon \\ &\quad \times \left\{\int_0^t \|A^{\alpha+\varepsilon}T(s)\|_{X\to Y}\,ds\right\} \times \left\{\sup_{0\le t<T_{\max}} \|G(u(t))\|_X\right\} \\ &\quad + \left\{\int_\tau^t \|A^\alpha T(s)\|_{X\to Y}\,ds\right\} \times \left\{\sup_{0\le t<T_{\max}} \|G(u(t))\|_X\right\}. \end{aligned}$$

Thus we have $\|u(t)-u(\tau)\|_Y = 0$ as $t, \tau \to T_{\max}$, contradicting the maximality of $T_{\max}$.

If $\lim_{t\to T_{\max}} \|u(t)\|_Y \ne \infty$ then there exist numbers $r > 0$ and $d > 0$ with d arbitrarily large and sequences $\tau_n \to T_{\max}$, $t_n \to T_{\max}$ as $n \to \infty$ and such that $\tau_n < t_n < T_{\max}$, $\|u(\tau_n)\|_Y = r$, $\|u(t_n)\|_Y = r + d$ and $\|u(t)\|_Y \le r + d$ for $t \in [\tau_n, t_n]$. We have

$$\begin{aligned} \|u(t_n)-u(\tau_n)\|_Y &\le \|[T(t_n-\tau_n)-I]u(\tau_n)\|_Y \\ &\quad + \left\|\int_{\tau_n}^{t_n} A^\alpha T(t_n-s)G(u(s))\,ds\right\|_Y. \end{aligned} \tag{13}$$

The left side of (13) is bounded below by $d > 0$ while the right side of (13) approaches zero as τ_n and t_n approach $T_{\max}$. This contradiction gives the result and (11) follows from (12).

4. Application to partial differential equations

In this section we will apply the results of §3 to a generalized Burger equation,

$$u_t(x,t) - \Delta u(x,t) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\{u(x,t)|u(x,t)|^{\gamma_i-1}\}, \qquad (x,t) \in \Omega \times (0, T), \tag{14}$$

$$u(x,0) = \phi(x), \qquad x \in \Omega;\ \phi(x) \in L^{p\gamma}(\Omega), \tag{15}$$

$$u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0, T); \tag{16}$$

to a nonlinear Cahn-Hilliard equation used in modelling the dynamics of pattern formation in phase transition,

(17) $u_t(x, t)+\Delta^2 u(x, t)+\Delta[u(x, t)-u^3(x, t)]=0, \qquad (x, t)\in\Omega\times(0, T),$

(18) $$u(x, 0)=\phi(x), \qquad x\in\Omega;\ \phi(x)\in L^{3p}(\Omega),$$

(19) $$\frac{\partial u(x, t)}{\partial\nu}=\frac{\partial\Delta u(x, t)}{\partial\nu}=0, \qquad (x, t)\in\partial\Omega\times(0, T),$$

ν the outward unit normal to $\partial\Omega$; to a Navier-Stokes equation

(20) $u_t(x, t)-\nu\Delta u(x, t)+(u\cdot\nabla)u+\nabla p=0, \qquad (x, t)\in\Omega\times(0, T),$

(21) $$\operatorname{div} u(x, t)=0, \qquad (x, t)\in\Omega\times(0, T),$$

(22) $$u(x, t)=0, \quad \text{on } \partial\Omega\times(0, T),$$

(23) $$u(x, 0)=\phi(x), \qquad x\in\Omega;\ \phi(x)\in L^{2p}(\Omega).$$

In order to apply the results of §3 to these concrete equations, it is necessary to demonstrate the conditions (i) through (vii) and the factorization mentioned in §1. In order to accomplish this, we must have a reasonable characterization of the domain of A^α for our generator A.

Let $1<p<\infty$ and let $\Omega\subset R^n$ be a bounded, open, connected subset with smooth boundary $\partial\Omega$. Let $X=L^p(\Omega)$ be equipped with the usual norm

$$\|u\|_p=\left(\int_\Omega |u(x)|^p\,dx\right)^{1/p}$$

and denote by $W^{m,p}(\Omega)$ the Sobolev space of all functions on Ω whose distributional derivatives up through order m are in $L^p(\Omega)$ with norm given by

$$\|\phi\|_{m,p}=\left(\sum_{\alpha\le m}\left\|\frac{\partial^\alpha\phi}{\partial x}\right\|_p^p\right)^{1/p}.$$

Let $A_p=-\Delta$ denote the negative Laplacian in $L^p(\Omega)$ with Dirichlet boundary conditions:

$$D(A_p)=W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega).$$

By [7, Theorem 3.5, p. 214] we know that $-A_p$ is the infinitesimal generator of an analytic semigroup $\{T(t): t\ge 0\}$ in $L^p(\Omega)$ and the fractional power $A_p^{1/2}$ is well defined. Following the notation of §2 we let $X_{1/2}$ denote the domain of $A_p^{1/2}$ equipped with the graph norm. In the special case $p=2$, $X_{1/2}=H_0^1(\Omega)$, D. Henry [5, p. 29]. When $p\ne 2$ and $\Omega=R^n$, $X_{1/2}=W^{1,p}(R^n)$. In the general case when Ω is bounded, the combined works of various authors are summarized in Treibel [9] to prove that

(24) $$D[(A_p+\lambda)^{1/2})]=W_0^{1,p}(\Omega), \qquad \lambda\ge 0,$$

with equivalent norms (see especially the proof of Theorem 4.9.2, p. 335). Taking $\lambda=0$ gives that $X_{1/2}=W_0^{1,p}(\Omega)$. We shall use this result to prove the following factorization theorem. In the following, let $D_j=\partial/\partial x_j$ denote the first order partial derivative with respect to x_j in the sense of distributions. The following theorem is proved in Heard and Rankin [6] and is proved again here for completeness.

Theorem 4. *Let $1 < p < \infty$ and $\Omega \subset R^n$ be a bounded, open connected subset with smooth boundary $\partial\Omega$. Let A_p denote the negative Laplacian in $L^p(\Omega)$ with Dirichlet boundary conditions. Then there is a bounded linear operator $B_j\colon L^p(\Omega) \to L^p(\Omega)$ such that $B_j(W^{1,p}(\Omega)) \subseteq W_0^{1,p}(\Omega)$ and*

$$D_j = A_p^{1/2} B_j, \quad \text{on } W^{1,p}(\Omega). \tag{25}$$

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$ and define $\widetilde{B}_j$ on $L^q(\Omega)$ by

$$\widetilde{B}_j = D_j \circ A_q^{-1/2}.$$

By (24), $A_q^{-1/2}$ is a continuous linear mapping from $L^q(\Omega)$ onto $W_0^{1,q}(\Omega)$. So $\widetilde{B}_j$ is well defined on all of $L^q(\Omega)$ and is a bounded linear operator from $L^q(\Omega)$ into itself. Thus $\widetilde{B}_j$ has a well-defined adjoint $\widetilde{B}_j^*$ which is bounded from $L^p(\Omega)$ into itself. Let $\langle\ ,\ \rangle$ denote the natural pairing between $L^p(\Omega)$ and $L^q(\Omega)$:

$$\begin{aligned} &\langle f, g\rangle = \int_\Omega f(x)g(x)\,dx, \qquad f \in L^p(\Omega),\ g \in L^q(\Omega). \\ &\langle \widetilde{B}_j^* f, g\rangle = \langle f, \widetilde{B}_j g\rangle, \quad \text{for all } f \in L^p(\Omega),\ g \in L^q(\Omega). \end{aligned} \tag{26}$$

Now if $f \in C_0^\infty(\Omega)$, then the definition of D_j gives that

$$\langle f, \widetilde{B}_j g\rangle = -\langle (A_q^{-1/2})^* D_j f, g\rangle, \quad \text{for all } g \in L^q(\Omega). \tag{27}$$

So from (25) and (26) we obtain that

$$\widetilde{B}_j^* f = (A_q^{-1/2})^* D_j f \tag{28}$$

for all $f \in C_0^\infty$. Taking limits we obtain (27) for all f in $W_0^{1,p}(\Omega)$. Now by [7, Lemma 3.4, p. 213] we have $A_q^* = A_p$ and by [11, Theorem 2, p. 225] we have $(R(\lambda; A_q))^* = R(\lambda; A_p)$ for all $\lambda \geq 0$. So by [7, formula (6.4), p. 69] we have $(A_q^{-1/2})^* = A_p^{-1/2}$ in $L^p(\Omega)$ and therefore by (27) we have

$$\widetilde{B}_j^* f = -A_p^{-1/2} D_j f, \quad \text{for all } f \in W_0^{1,p}(\Omega).$$

Hence $\widetilde{B}_j^*\colon W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ continuously and

$$A_p^{1/2} \widetilde{B}_j^* = -D_j f, \quad \text{for all } f \in W_0^{1,p}(\Omega).$$

Setting $B_j = -\widetilde{B}_j^*$ we obtain

$$B_j f = A_p^{-1/2} D_j f, \quad \text{for all } f \in W_0^{1,p}(\Omega).$$

The operator $f \to A_p^{-1/2} D_j f$ is well defined and bounded from $W^{1,p}(\Omega)$ into $W_0^{1,p}(\Omega)$. Therefore B_j represents the continuous linear extension of $A_p^{-1/2} D_j$ for, $W^{1,p}(\Omega)$ to all of $L^p(\Omega)$.

The following lemma is proved in [10, Lemma 4.1, p. 293].

Lemma 4. *Suppose $-A$ generates an analytic semigroup $T(t)$ on each $L^p(\Omega)$ space. Further, suppose there exists a positive integer m such that for each p, $D(A_p)$ with its graph norm is continuously embedded in $W^{m,p}(\Omega)$ where the boundary of Ω is of class C^m. Then $T(t)\colon L^p(\Omega) \to L^q(\Omega)$ for $1 < p \leq q < \infty$*

and $t > 0$ is a bounded operator. Furthermore, for any $T > 0$ there is a constant C depending on p and q such that

$$\|T(t)\phi\|_q \leq Ct^{-n/mr}\|\phi\|_p \tag{29}$$

for all $t \in (0, T]$, and $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$.

From (29) and the fact that $A_p^\alpha T(t) = T(\frac{t}{2})A_p^\alpha T(\frac{t}{2})$ for $0 \leq \alpha \leq 1$, we deduce the following corollary to Lemma 4.

Corollary 5. *Suppose the assumptions of Lemma 4 hold, then for $t \in (0, T]$*

$$\|A_p^\alpha T(t)\phi\|_q \leq Ct^{-(n/mr+\alpha)}\|\phi\|_p \tag{30}$$

where $0 \leq \alpha \leq 1$ and C again depends on p and q.

We can now prove the following theorem.

Theorem 5. *Suppose $1 \leq n < p < \infty$ and $\gamma = \max\{\gamma_i: 1 \leq i \leq n\}$, then for each $\phi \in L^{p\gamma}(\Omega)$, there exists a unique global strong solution of equation (14)(15)(16).*

Proof. Define the mapping h_i on $L^{p\gamma_i}(\Omega)$ by $h_i(u(x)) = u(x)|u(x)|^{\gamma_i - 1}$ for $\gamma_i > 1$. Let $\gamma = \max\{1 \leq i \leq n\}$, then if B_i is as in Theorem 4, the mapping defined by $g_i = B_i h_i$ takes $L^{p\gamma}(\Omega)$ into $L^p(\Omega)$ and satisfies the following local Lipschitz condition:

$$\|g_i(u(x)) - g_i(v(x))\|_p \leq M_i(R)\|u(x) - v(x)\|_{p\gamma} \tag{31}$$

for $\|u(x)\|_{p\gamma}$, $\|v(x)\|_{p\gamma} \leq R$. Now defining $G(u(x)) = \sum_{i=1}^n g_i(u(x))$ we have that $G: L^{p\gamma}(\Omega) \to L^p(\Omega)$ and for u, v satisfying $\|u(x)\|_{p\gamma}, \|v(x)\|_{p\gamma} \leq R$, $\|G(u(x)) - G(v(x))\|_p \leq M(R)\|u(x) - v(x)\|_{p\gamma}$. Here $M(R)$ depends on $M_i(R)$ for $1 \leq i \leq n$. The function

$$F(u(x)) = A_p^{1/2}G(u(x)) \tag{32}$$

is well defined as a mapping from $W^{1,p}(\Omega)$ into $L^p(\Omega)$ and for $u \in W_0^{1,p}(\Omega)$

$$F(u(x)) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\{h_i(u(x))\} \tag{33}$$

where the derivatives are taken in the generalized sense. Thus for u, v in $W_0^{1,p}(\Omega)$ such that $\|u(x)\|_{1,p}$, $\|v(x)\|_{1,p} \leq R$ we have for $n < p < \infty$ by the Sobolev Imbedding Theorem

$$\begin{aligned} \|F(u(x)) - F(v(x))\|_p &\leq K(R)\|u(x) - v(x)\|_{1,p} \\ &\leq K(R)\|A_p^{1/2}(u(x) - v(x))\|_p. \end{aligned} \tag{34}$$

Now from Lemma 4, Corollary 5, and (31) through (35), we see that the conditions of §2 are satisfied as well as the assumptions of the lemmas and theorems of §3. Thus by Theorem 2 we have the existence of a local strong solution of equation (14)(15)(16). In order to show that this solution is a global strong solution we multiply equation (15) by $|u(x, t)|^{r-1}\operatorname{sign} u(x, t)$ and integrate to obtain

$$\frac{\partial(\frac{1}{r}\int_\Omega |u(x, t)|^r\, dx)}{\partial t} + \int_\Omega |\nabla u^{r/2}(x, t)|^2\, dx = 0. \tag{35}$$

Since $u(x, t)$ is in $W_0^{1,p}$ for $t > 0$, for $r > p > 1$ we have by the Poincaré inequality and equation (35) that

$$\frac{\partial(\frac{1}{r}\int_\Omega |u(x, t)|^r\,dx)}{\partial t} + C(\Omega)\int_\Omega |u(x, t)|^r\,dx \leq 0. \tag{36}$$

For $r > p$ we see from (36) that

$$\int_\Omega |u(x, t)|^r\,dx \leq e^{-pC(\Omega)t}\left(\int_\Omega |\phi(x)|^r\,dx\right).$$

Letting $r = p\gamma$ we have that a solution of equation (14)(15)(16) is bounded in the $L^{p\gamma}(\Omega)$ norm, uniformly in t, and so by Theorem 3 the mild solution of equation (14)(15)(16) exists for all time. However appealing to Theorem 2 we see that the strong solution of equation (14)(15)(16) exists for all $t > 0$.

Here it is emphasized again that the theory of §3 allows us to obtain *global strong solutions of equation* (14)(15)(16) *starting from initial data which do not have derivatives.* This is an improvement over existing abstract theory for semi-linear equations.

Theorem 6. *Suppose that* $n = 3$ *and* $p > n$*, then for each* $\phi \in L^{3p}(\Omega)$ *there exists a unique global solution of the Cahn-Hilliard equation* (17)(18)(19).

Proof. For equation (17)(18)(19) $A_p = \Delta^2$ with domain $D(A_p) = \{\phi \in W^{4,p}(\Omega): \partial\phi/\partial\nu = \partial\Delta\phi/\partial\nu = 0 \text{ on } \partial\Omega\}$, ν the outward unit normal on $\partial\Omega$. In this case $A_p^{1/2} = -\Delta$ and the domain $D(A_p^{1/2}) = \{\phi \in W^{2,p}(\Omega): \partial\phi/\partial\nu = 0\}$. We have that F maps $D(A_p^{1/2})$ into $L^p(\Omega)$ and G defined by $G(\phi) = -\phi + \phi^3$ maps $L^{3p}(\Omega)$ into $L^p(\Omega)$. It is easy to see that G is locally Lipschitz and if $p > n$ so is F. Furthermore, for $\phi \in D(A_p^{1/2})$ we have $F(\phi(x)) = -\Delta G(\phi(x)) = A_p^{1/2}G(\phi(x))$. From this discussion and the application of Lemma 4 and Corollary 5 to the operator $A_p = \Delta^2$ we see that the conditions of §2 are met as well as the assumptions of Theorems 1 and 2 of §3. Thus applying these results to equation (17)(18)(19) we obtain the local existence of a *strong solution* for initial data from $L^{3p}(\Omega)$. Since by Theorem 2 this local solution is in $D(A_p)$ for $t > 0$ we can, following Temam [8, p. 156], obtain an a priori bound for $\|\Delta u(x, t)\|_2$. Now since $W^{2,2}(\Omega)$ is embedded in $C^{1/2}(\overline{\Omega})$ we have that $\|u(x, t)\|_{3p}$ is bounded uniformly in t. By Theorem 3 the mild solution guaranteed by Lemma 3 is global. However appealing to Theorem 2 we see that $u(x, t)$ is actually a global strong solution. Once again we have illustrated that the theory developed in §3 gives a *strong global solution* of equation (17)(18)(19) for initial data in $L^{3p}(\Omega)$.

The final example used to illustrate the abstract theory developed in this paper is the Navier-Stokes equation (20)(21)(22)(23). We define $X^p = (L^p(\Omega))^n$ and let the function $u(x, t) = (u^1(x, t), \dots, u^n(x, t))$. Set

$$E^p = \text{closure in } X^p \text{ of } \{\phi \in (C_0^\infty(\Omega))^n;\ \operatorname{div}\phi(x) = 0\},$$
$$J_p = \{\nabla p;\ p \in W^{1,p}(\Omega)\}.$$

We have the following well-known decomposition [4],

$$X^p = E^p \oplus J_p.$$

Theorem 7. *If* $1 \leq n < p < \infty$ *then for each* $\phi \in E^{2p}$ *there exists a unique strong local solution of the Navier-Stokes equation* (20)(21)(22)(23).

Proof. Denote by P the continuous projection from X^p to E^p and let Δ_p be to Laplace operator with $D(\Delta_p) = \{\phi \in (W^{2,p}(\Omega))^n ; \phi(x) = 0, x \in \partial\Omega\}$. Define $A_p = -P\nu\Delta_p$ with domain $D(A_p) = E_p \cap D(\Delta_p)$. It is known [1] that $-A_p$ is a closed densely defined linear operator in E^p with a bounded inverse and generates a bounded analytic C_0 semigroup in E^p.

For $u(x, t)$ in $(W_0^{1,p}(\Omega))^n \cap E^p$ we have

$$\left(\sum_{k=1}^n u^k(x, t)\frac{\partial u^1(x, t)}{\partial x_k}, \ldots, \sum_{k=1}^n u^k(x, t)\frac{\partial u^n(x, t)}{\partial x_k}\right) = \left(\sum_{k=1}^n \frac{\partial\{u^k(x, t)u^1(x, t)\}}{\partial x_k}, \ldots, \sum_{k=1}^n \frac{\partial\{u^k(x, t)u^n(x, t)\}}{\partial x_k}\right).$$

Applying the projection P to equation (20) we have the following abstract formulation of equation (20)(21)(22)(23),

$$u(t) + A_p u(t) = F(u(t)), \qquad t > 0, \tag{37}$$

$$u(0) = \phi. \tag{38}$$

Here F is defined on $D(A_p^{1/2}) = (W_0^{1,p}(\Omega))^n \cap E^p$ with range in E^p by

$$F(\phi(x)) = P\sum_{k=1}^n \left(\frac{\partial\{\phi^k(x)\phi^1(x)\}}{\partial x_k}, \ldots, \frac{\partial\{\phi^k(x)\phi^n(x)\}}{\partial x_k}\right).$$

If $p > n$ and using the fact that $(W^{1,p}(\Omega))^n$ is a Banach algebra it is easy to show that F is locally Lipschitz from $D(A_p^{1/2})$ into E^p. In a recent paper of Bridges [2] the results of §3 are applied to a Navier-Stokes equation. In [2] it is shown that there exists a B_k such that B_k is bounded from X^p into E^p with $P\ \partial/\partial x_k = A_p^{1/2}B_k$. It is further shown in [2] that the mapping

$$G(\phi(x)) = \sum_{k=1}^n B_k(\phi^k(x)\phi^1(x), \ldots, \phi^k(x)\phi^n(x))$$

maps E^{2p} into E^p and is locally Lipschitz continuous; that the analytic semigroup $T(t)$, $t \geq 0$ generated by A_p satisfies $A_p^{1/2}T(t): E^p \to E^q$ for $t > 0$ and $1 < p < q < \infty$ and $\|A_p^{1/2}T(t)\phi\|_q \leq Ct^{-(z+1/2)}\|\phi\|_p$ where $z = n(\frac{1}{p} - \frac{1}{q})$ and C depends on p and q. If $q = 2p$ and $p > n$ then $z + \frac{1}{2}$ is less than 1. Thus the theory developed in §3 applies and equation (37)(38) is guaranteed a unique local *strong solution*.

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